On the Clebsch-Gordan coefficients for the two-parameter quantum algebra $SU(2)_{p,q}$

1993 J. Phys. A: Math. Gen. 26 5177

(http://iopscience.iop.org/0305-4470/26/19/045)

J. Phys. A: Math. Gen. **26** (1993) 5177–5179. Printed in the UK

COMMENT

On the Clebsch–Gordan coefficients for the two-parameter quantum algebra $SU(2)_{p,q}$

Stjepan Meljanac† and Marijan Milekovi醇

† Rudjer Bošković Institute, Bijenička c. 54, 41001 Zagreb, Croatia
‡ Prirodoslovno–Matematički fakultet, Department of Theoretical Physics, Bijenička c. 32, 41000 Zagreb, Croatia

Received 5 March 1993

Abstract. We show that the Clebsch–Gordan coefficients for the $SU(2)_{p,q}$ algebra depend on a single parameter $Q=\sqrt{pq}$, contrary to the explicit calculation of Smirnov and Wehrhahn.

Recently, the Clebsch–Gordan problem for the two-parameter quantum algebra $SU(2)_{p,q}$ was analysed [Smirnov and Wehrhahn 1992]. It was claimed that the corresponding Clebsch–Gordan coefficients do depend on the two deforming parameters p and q.

In this comment we show that the Clebsch–Gordan coefficients depend effectively only on one parameter $Q=\sqrt{pq}$, and that $SU(2)_{p,q}$ is isomorphic to $SU(2)_Q$, both as algebras and Hopf co-algebras. Our results are in agreement with Drinfeld (1989).

We recall the $SU(2)_{p,q}$ algebra defined by Smirnov and Wehrhahn (1992) (p and q are real parameters):

$$
\begin{aligned}
&[J_0, J_\pm] = \pm J_\pm \\
&[J_+, J_-]_{p,q} = J_+J_- - pq^{-1}J_-J_+ = [2J_0]_{p,q} \\
&[2J_0]_{p,q} = \frac{q^{2J_0} - p^{-2J_0}}{q - p^{-1}} \\
&(J_0)^+ = J_0 \quad (J_\pm)^+ = J_\mp .
\end{aligned}
\tag{1}
$$

The coproduct Δ is:

$$
\begin{aligned}
&\Delta(J_\pm) = J_\pm \otimes p^{-J_0} + q^{J_0} \otimes J_\pm \\
&\Delta(J_0) = J_0 \otimes 1 + 1 \otimes J_0 .
\end{aligned}
\tag{2}
$$

The finite dimensional unitary irreducible representation (IRREP) D^j of spin j contains the highest weight vector $|jj\rangle$, satisfying

$$
\begin{aligned}
&J_0|jj\rangle = j|jj\rangle \\
&J_+|jj\rangle = 0 \\
&\langle jj|jj\rangle = 1 .
\end{aligned}
\tag{3}
$$

The other orthonormalized states of IRREP D^j, $|jm\rangle$, with $-j\leq m\leq j$, satisfy

$$J_+|jm\rangle_{p,q}=(pq^{-1})^{1/2(j-m-1)}\sqrt{[j-m]_{p,q}[j+m+1]_{p,q}}|jm+1\rangle_{p,q}$$
$$J_-|jm\rangle_{p,q}=(pq^{-1})^{1/2(j-m)}\sqrt{[j+m]_{p,q}[j-m+1]_{p,q}}|jm-1\rangle_{p,q}$$
$$J_0|jm\rangle_{p,q}=m|jm\rangle_{p,q}. \tag{4}$$

Now, we define the $SU(2)_Q$ algebra with three generators $(J_\pm)_Q$ and $(J_0)_Q$:

$$[(J_0)_Q,(J_\pm)_Q]=\pm(J_\pm)_Q$$
$$[(J_+)_Q,(J_-)_Q]=[2(J_0)_Q]_Q$$
$$[n]_Q=\frac{Q^n-Q^{-n}}{Q-Q^{-1}}=(p/q)^{1/2(n-1)}[n]_{p,q} \tag{5}$$

with the coproduct

$$\Delta((J_\pm)_Q)=(J_\pm)_Q\otimes Q^{-(J_0)_Q}+Q^{+(J_0)_Q}\otimes(J_\pm)_Q$$
$$\Delta((J_0)_Q)=(J_0)_Q\otimes 1+1\otimes(J_0)_Q.$$

The relations between the $SU(2)_{p,q}$ generators and the $SU(2)_Q$ generators are

$$J_+=(q/p)^{1/2(J_0-1/2)}(J_+)_Q$$
$$J_-=(q/p)^{1/2(J_0+1/2)}(J_-)_Q$$
$$J_0=(J_0)_Q. \tag{7}$$

It is easy to show that relations (7) map equations (1) and equations (5) one into another. Moreover, the $SU(2)_{p,q}$ coproduct is identical to the $SU(2)_Q$ coproduct, $\Delta_{p,q}\equiv\Delta_Q$:

$$\Delta_{p,q}(J_0)=\Delta_Q(J_0)=J_0\otimes 1+1\otimes J_0$$
$$\Delta_{p,q}(J_\pm)=\Delta_Q(J_\pm)=\Delta_Q((q/p)^{1/2(J_0\mp 1/2)}(J_\pm)_Q)$$
$$=J_\pm\otimes p^{-J_0}+q^{J_0}\otimes J_\pm. \tag{8}$$

This is also true for the antipode $\gamma_{p,q}\equiv\gamma_Q$, the counit $\varepsilon_{p,q}\equiv\varepsilon_Q$, the states $|jm\rangle_{p,q}\equiv|jm\rangle_Q$ and the Casimir operator $(C_2)_{p,q}\equiv(C_2)_Q$. Thus we have proved the Hopf-algebra isomorphism between $SU(2)_{p,q}$ and $SU(2)_Q$. As a consequence of this isomorphism, the p,q Clebsch–Gordan coefficients of $SU(2)_{p,q}$ should be identical to those of $SU(2)_Q$.

Returning to the Smirnov and Wehrhahn paper, one can immediately show, using our equation (5) and $[n]_{p,q}=(q/p)^{(n-1)/2}[n]_Q$, that all the equations in section 2 of their paper can be reduced to the equations with single parameter Q. Particularly, the states in equation (2.7) can be written as $|jm\rangle_{p,q}=|jm\rangle_Q$. Therefore, the projection operator $P^j_{mm'}=|jm\rangle\langle jm'|$ also has to be expressed in terms of the Q-parameter only. However, their projection operator, eq. (3.7), is wrong since it explicitly depends on the (p/q) parameter and does not satisfy the projection property $(P^j_{mm})^2=P^j_{mm}$. Hence, all the subsequent formulae in section 5 (starting with the equation (5.3)) are incorrect. For example, the Clebsch–Gordan coefficients given in table 1 do not satisfy the orthonormality relations (4.10).

The correct formula for the projection operator should read

$$
\begin{aligned}
p^j_{mm'} = \left(\frac{p}{q}\right)^{-1/4(j-m)(j-m-1)} & \sqrt{\frac{[j+m]_{p,q}!}{[2j]_{p,q}![j-m]_{p,q}!}} J_-^{j-m} P^j J_+^{j-m'} \\
& \times \left(\frac{p}{q}\right)^{-1/4(j-m')(j-m'-1)} \sqrt{\frac{[j+m']_{p,q}!}{[2j]_{p,q}![j-m']_{p,q}!}} \\
& \equiv (P^j_{mm'})_Q.
\end{aligned} \tag{9}
$$

Using this projector we obtain the Clebsch–Gordan coefficients which depend on the single parameter Q. In this way, these Clebsch–Gordan coefficients are identical to those for $SU(2)_Q$:

$$
\langle j_1 m_1 j_2 m_2 | JM \rangle_{p,q} = \langle j_1 m_1 j_2 m_2 | JM \rangle_{Q=\sqrt{pq}} \tag{10}
$$

for all j_1, m_1, j_2, m_2, J and M.

Acknowledgments

This work was supported by the joint Croatian–American contract NSF JF 999 and the Scientific Fund of Republic of Croatia.

References

Drinfeld V G 1989 *Algebra i Analiz* **1** 1 (in Russian)
Smirnov Yu F and Wehrhahn R F 1992 *J. Phys. A: Math. Gen.* **25** 5563